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A CHARACTERIZATION OF CLIQUE GRAPHS

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PREFACE

In this Memorandum we solve the graph-theoretical problem of characterizing clique graphs. The body of the Memorandum divides into two parts. The first part, consisting of pages 1-11, contains the main results and several principal applications of the main theorems. An abridged version consisting of this first part only has been submitted for publication in the Journal of Combinatorial Theory and also prepared for presentation at the American Mathematical Society Meeting in New York, April 4, 5, 1969. The second part of the Memorandum consists of additional applications of the main theorems, some partial results, and some questions posed for further research.

The results in this Memorandum have possible application in genetics and in the theory of measurement in psychology, through their connection with the interval graph research of Fulkerson and Gross (Incidence Matrices with the Consecutive Ones Property, RM-3984-PR). For the measurement applications of interval graphs, the reader is referred to RM-5782-PR, On Nontransitive Indifference (Roberts). There is also a potential sociological application—the graph-theoretical term "clique" arises from the corresponding sociological notion.

ABSTRACT

In a recent paper [4], Hamelink obtains an interesting sufficient condition for a graph to be a clique graph. In this Memorandum we give related conditions which are necessary as well as sufficient. As an application of our result we show that Hamelink's condition is also necessary in certain special cases and that here it can be greatly simplified. As another application, we derive certain theorems useful in practice in reducing the question of whether a given graph is a clique graph to whether certain smaller or simpler graphs are. Next, we relate the clique graph results to some work of Fulkerson and Gross [2] on interval graphs and the consecutive ones property. Finally, we include some remarks, motivated by our clique graph results, on graphs with no independent cut sets.

CONTENTS

A CHARACTERIZATION OF CLIQUE GRAPHS*

1. INTRODUCTION

Our graphs will all be finite, nondirected, with no loops or multiple edges. If G is a graph, V(G) will denote the set of vertices of G and E(G) the set of edges. We denote the adjacency relation by I, i.e., if $x, y \in V(G)$, then xIy iff $(x, y) \in E(G)$. A clique of G is a maximal complete subgraph. (Some authors use the terminology dominant clique.) Given G, let $K_1, K_2, \ldots, K_n$ be its cliques. Define H by $V(H) = \{K_1, K_2, \ldots, K_n\}$ and $(K_i, K_j) \in E(H)$ iff $i \neq j$ and $K_i \cap K_j \neq \emptyset$. Then we call H the clique graph of G and write $H = K(G)$. The main problem we are concerned with is this: given a graph H, is it the clique graph of some G?

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*The authors would like to acknowledge the helpful comments of Jon Folkman and Ray Fulkerson.

2. THE CHARACTERIZATION

Let $\mathcal{K}$ be a collection of complete subgraphs of a graph H. We shall say $\mathcal{K}$ has property $\mathcal{J}$ (for intersection) if whenever $L_1, L_2, \ldots, L_p$ are in $\mathcal{K}$ and $L_i \cap L_j \neq \emptyset$ for all i, j then the total intersection $\bigcap_{i=1}^{p} L_i \neq \emptyset$. We say $\mathcal{K}$ has property $\mathcal{J}_m$ if the above holds whenever p = m. Finally, let $\mathcal{K}(H)$ be the collection of all cliques of the graph H.

THEOREM 1. (Hamelink): If $\mathcal{K}(H)$ satisfies property $\mathcal{J}$ then H is a clique graph.

Note how the condition that the points of H represent cliques is reflected in the cliques of H itself. The converse of Theorem 1 is not true. To give an example, let H and G be the graphs shown in Fig. 1. Then H = K(G), but the set $\mathcal{K}(H)$ does not satisfy property $\mathcal{J}$. For, take

$$L_1 = \{A, B, C, D\},\ L_2 = \{E, B, F, G\} \text{ and } L_3 = \{I, D, G, H\}.$$

THEOREM 2. (Characterization of Clique Graphs): A graph H is a clique graph iff there is a collection $\mathcal{K}$ of complete subgraphs of H which satisfies the following two properties:

(1) $\mathcal{K}$ covers all the edges of H, i.e., if $x, y \in H$ and xIy, then $\{x, y\}$ is contained in some element of $\mathcal{K}$.

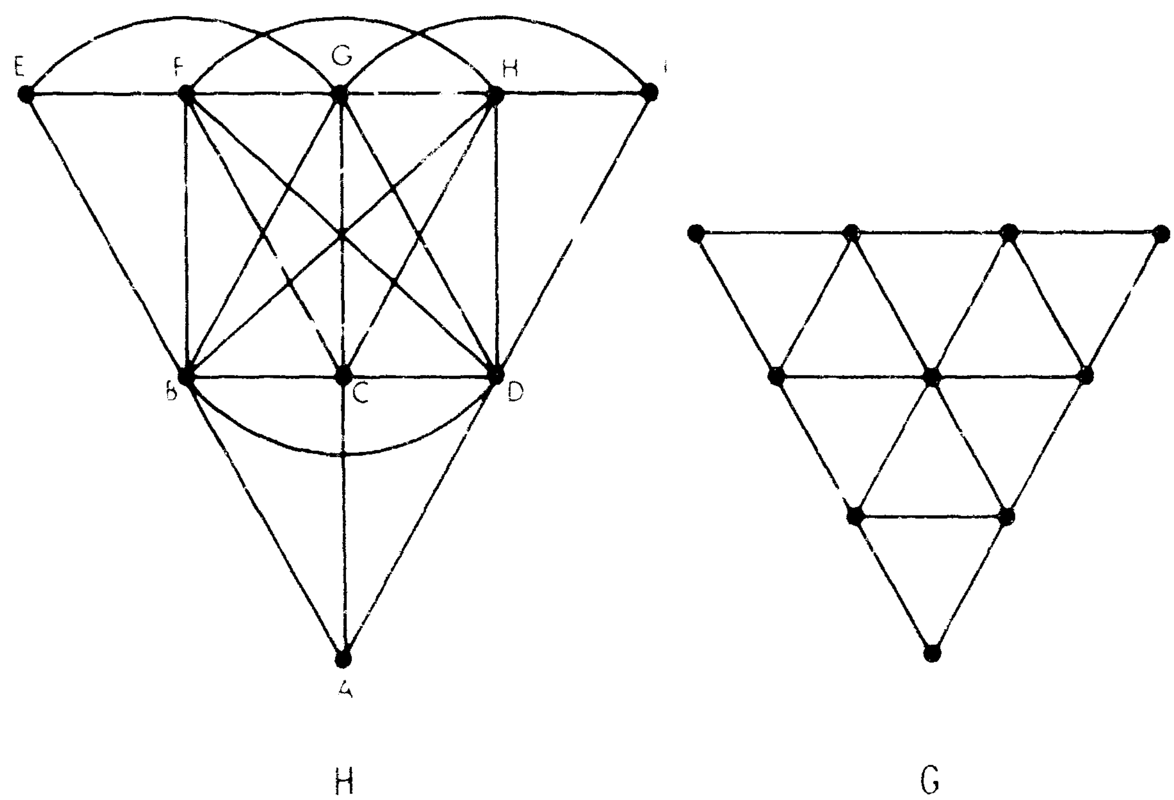

Figure 1

(2) $\mathcal{K}$ satisfies property $\mathcal{J}$.

Proof. The proof of sufficiency is essentially Hamelink's proof of Theorem 1. Let $\mathcal{K} = \{L_1, L_2, \ldots, L_p\}$. Define the graph G as follows.

$V(G) = V(H) \cup \mathcal{K}$

If $h \in V(H)$, then $h \, I \, L_i$ iff $h \in L_i$

$L_i \, I \, L_j$ iff $i \neq j$ and $L_i \cap L_j \neq \emptyset$

If $h, h' \in V(H)$, then not hIh'.

The claim is that $H = K(G)$. To prove this, let $C(h) = \{h\} \cup \{L_i : h \in L_i\}$. It is easy to see that each $C(h)$ is a clique of G. Moreover, these are the only cliques of G. For, let C be a complete subgraph of G. Then if C contains an element h of V(H), we have $C \subseteq C(h)$. And otherwise, C is contained in some C(h) by property $\mathcal{J}$.

To prove the necessity of the conditions, suppose $H = K(G)$. Let $V(G) = \{g_1, g_2, \ldots, g_n\}$, let $V(H) = \{h_1, h_2, \ldots, h_m\}$, and let $K_1, K_2, \ldots, K_m$ denote the cliques of G, labelled in such a way that $h_i I h_j$ iff $K_i \cap K_j \neq \emptyset$. For $i = 1, 2, \ldots, n$, define $L_i = \{h_j : g_i \in K_j\}$. Each L_i is complete, because if h_j and h_k are in L_i, then $g_i \in K_j \cap K_k$ and so $h_j I h_k$. The claim is that $\mathcal{K} = \{L_1, L_2, \ldots, L_n\}$ satisfies properties (1) and (2). Property (1) is satisfied because if $h_j I h_k$ then $K_j \cap K_k \neq \emptyset$. Finally, $\mathcal{K}$ satisfies property $\mathcal{J}$. For, suppose $L_{i_1}, L_{i_2}, \ldots, L_{i_r}$ pairwise intersect. Then for all j, k, there

is a point h_{jk} in $L_{i_j} \cap L_{i_k}$. Thus g_{i_j} and g_{i_k} are both in K_{jk} and therefore we have $g_{i_j} I g_{i_k}$. It follows that $\{g_{i_1}, g_{i_2}, \ldots, g_{i_r}\}$ is contained in some clique K_s of G and thus $h_s \in \bigcap_{t=1}^{r} L_{i_t}$. Q.E.D.

Remark. Theorem 2 is reminiscent of Krausz' [5] characterization of line graphs.

3. THE CASE OF CLIQUE NUMBER ≤ 3

There are certain situations where the conditions of Theorem 2 may be simplified, i.e., where the conditions of Hamelink become necessary as well as sufficient. This fact will follow by a simple application of Theorem 2. We first require one lemma.

LEMMA 1. Suppose $\mathcal{K}$ is a collection of complete subgraphs of a graph H, $\mathcal{K}$ satisfies (1) and (2) of Theorem 2, and suppose no member of $\mathcal{K}$ is contained in any other. Then $\mathcal{K}$ contains a 2-element set iff this set is a clique of H.

Proof. Every 2-element clique is contained in $\mathcal{K}$ by property (1). Conversely, suppose $L_1 = \{h, h'\} \in \mathcal{K}$ and there is a point $h'' \neq h, h'$ which is adjacent to both h and h'. Then there are sets L_2 and L_3 in $\mathcal{K}$ such that $\{h, h''\} \subseteq L_2$ and $\{h', h''\} \subseteq L_3$. It follows that L_1, L_2, L_3 pairwise intersect but have no point in common, violating property $\mathcal{J}$. Q.E.D.

Definition. $\omega(H)$ = clique number of H = $\max\{|L| : L \text{ is a clique of } H\}$.

THEOREM 3. If $\omega(H) \leq 3$, then H is a clique graph iff $\mathcal{K}(H)$ satisfies property $\mathcal{J}$.

Proof. If H is a clique graph then there is some collection $\mathcal{K}$ of complete subgraphs satisfying properties (1) and (2) of Theorem 2.

Let $\mathcal{K}'$ be the collection of all (setwise) maximal elements of $\mathcal{K}$ together with all one-element cliques of H. This collection still satisfies properties (1) and (2). We shall show that $\mathcal{K}' = \mathcal{K}(H)$. $\mathcal{K}' \subseteq \mathcal{K}(H)$ follows directly by Lemma 1 since $\omega(H) \leq 3$. To show $\mathcal{K}(H) \subseteq \mathcal{K}'$, suppose $L \in \mathcal{K}(H)$. That $L \in \mathcal{K}'$ follows easily if $|L| < 3$. Thus, let $L = \{h_1, h_2, h_3\}$. By property (1) $\mathcal{K}'$ has elements L_1, L_2, L_3 containing $\{h_1, h_2\}$, $\{h_1, h_3\}$ and $\{h_2, h_3\}$, respectively. Since $\mathcal{K}'$ satisfies property (2), there is a point h in $L_1 \cap L_2 \cap L_3$. Since h is in each L_i, it is adjacent to or equal to each point h_i. Thus $\{h_1, h_2, h_3, h\}$ is complete in H and $\omega(H) \leq 3$ implies that $h = h_i$, some i. If $i = 1$, say, then $L_3 = \{h_1, h_2, h_3\}$ and so $L \in \mathcal{K}'$!

The converse follows by Theorem 2. Q.E.D.

Actually it turns out that if $\omega(H) \leq 3$, property $\mathcal{J}$ is equivalent to the much weaker property $\mathcal{J}_3$. This will follow from the next lemma, and will give us a very simple criterion for clique graphs if $\omega(H) \leq 3$.

LEMMA 2. Suppose $\omega(H) \leq m$ and $\mathcal{K}$ is a collection of complete subgraphs of H. Then $\mathcal{K}$ satisfies property $\mathcal{J}$ iff it satisfies property $\mathcal{J}_m$.

Proof. The case $m = 1$ is trivial. Suppose $m > 1$ and suppose $\mathcal{K}$ satisfies $\mathcal{J}_m$ but not $\mathcal{J}$. Then there are $L_1, L_2, \ldots, L_p$ in $\mathcal{K}$ which pairwise intersect but have no point in common. We begin by observing that $|L_i \cap L_j| = m-1$, all $i \neq j$.

For $|L_i \cap L_j| < |L_i| \leq m$. Suppose $|L_i \cap L_j| = r < m-1$. Let $L_i \cap L_j = \{k_1, k_2, \ldots, k_r\}$. Then for each u there is L_{t_u} such that $k_u \notin L_{t_u}$. Hence $\{L_i, L_j, L_{t_1}, L_{t_2}, \ldots, L_{t_r}\}$ consists of $\leq m$ elements of $\mathcal{K}$ which pairwise intersect but have no point in common, violating property $\mathcal{J}_m$.

Since $|L_i \cap L_j| = m-1$, all $i \neq j$, it follows that there are distinct points $h_1, h_2, \ldots, h_{m+1}$ in H so that

$$L_i = \{h_1, h_2, \ldots, h_{i-1}, \hat{h}_i, h_{i+1}, \ldots, h_{m+1}\},$$

where the symbol $\hat{h}_i$ means h_i is omitted. But now the points h_j and h_k are adjacent in H for all $j \neq k$, because h_j, h_k are in the complete subgraph L_i for $i \neq j, k$. Thus $\{h_1, h_2, \ldots, h_{m+1}\}$ is a complete subgraph of H, and this violates $\omega(H) = m$. Q.E.D.

THEOREM 4. If $\omega(H) \leq 3$, then H is a clique graph iff $\mathcal{K}$(H) satisfies property $\mathcal{J}_3$.

Proof. Theorem 3 and Lemma 2. Q.E.D.

Definition. A graph H_1 is a partial subgraph of a graph H_2 if $V(H_1) \subseteq V(H_2)$ and $E(H_1) \subseteq E(H_2)$.

COROLLARY. If $\omega(H) \leq 3$, then H is a clique graph iff it has no partial subgraph isomorphic to the graph of Fig. 2.

Proof. Suppose H has such a partial subgraph. Since $\omega(H) \leq 3$, the three outer triangles are cliques. These pairwise intersect but have no point in common, violating property $\mathcal{J}_3$ for $\mathcal{K}(H)$. Conversely, suppose $\mathcal{K}(H)$ does not satisfy property $\mathcal{J}_3$. Let K_1, K_2, K_3 be three cliques which pairwise intersect but have no point in common. Using $\omega(H) \leq 3$, it is easy to prove that each K_i is a triangle. Moreover, $|K_i \cap K_j| = 1$, $i \neq j$. For suppose for example $|K_1 \cap K_2| = 2$. Let $K_1 = \{h_1, h_2, h_3\}$ and let $K_2 = \{h_1, h_2, h_4\}$. Then, since $K_1 \cap K_3 \neq \emptyset$, $K_2 \cap K_3 \neq \emptyset$ and $K_1 \cap K_2 \cap K_3 = \emptyset$, we conclude $K_3 = \{h_3, h_4, h_5\}$, some h_5. It follows that $\{h_1, h_2, h_3, h_4\}$ is complete, violating $\omega(H) \leq 3$. Thus, K_1, K_2, K_3 are triangles with no common point, each pair of which has exactly one point in common. This implies that the vertices of K_1, K_2, K_3 are the vertices of a partial subgraph isomorphic to the graph of Fig. 2. Q.E.D.

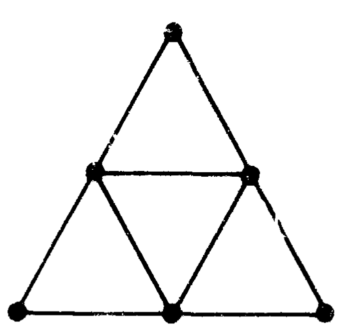

Figure 2

4. REDUCTION THEOREMS

As a further application of Theorem 2, we present some results which might be useful as tools in reducing the question of whether a given graph is a clique graph to whether certain smaller or simpler graphs are clique graphs. The proofs are straightforward using the characterization.

THEOREM 5. *Suppose* H *is disconnected and* $H_1, H_2, \ldots, H_p$ *are its components. Then* H *is a clique graph iff each* H_i *is.*

Proof. Trivial (even without the characterization).

THEOREM 6. *Suppose* H *is a connected graph with a cut point* h. *Let* $H - h = H_1' + H_2'$, $H_1' \cap H_2' = \emptyset$, *and suppose there is no edge from* H_1' *to* H_2'. If $H_i = H_i' + h$, *then* H *is a clique graph iff* H_1 *and* H_2 *are.*

Proof. If $\mathcal{K}_i$ is a collection of complete subgraphs of H_i satisfying (1) and (2), then $\mathcal{K} = \mathcal{K}_1 \cup \mathcal{K}_2$ satisfies (1) and (2) for H. Conversely if $\mathcal{K}$ is a collection of complete subgraphs of H satisfying (1) and (2), then $\mathcal{K}_i$ $\{L \in \mathcal{K} : L \subseteq V(H_i)\}$ is a collection of complete subgraphs satisfying (1) and (2) for H_i. Q. E. D.

COROLLARY 6.1. *Suppose* H *is a connected graph with a cut point* h. *Let* $H'_1, H'_2, \ldots, H'_n$ *be the components of* $H - h$ *and let* H_i *be the subgraph generated by* H'_i *plus the vertex* h. *Then* H *is a clique graph iff each* H_i *is.*

Proof. The argument is similar. Q.E.D.

Definition. Suppose H is a graph and S is a subset of V(H) so that $h, h' \in S$ implies not hIh'. Then S is an *independent set*. If in addition S is a cut set, S will be called an *independent cut set.*

COROLLARY 6.2. *Suppose* H *is a connected graph and* S *is an independent cut set of* H. *Let* $H - S = H'_1 + H'_2$, $H'_1 \cap H'_1 = \emptyset$ *and suppose that there is no edge from* H'_1 *to* H'_2. *If* H_i *is the subgraph of* H *generated by* H'_i *plus* S, *then* H *is a clique graph iff* H_1 *and* H_2 *are.*

Proof. The argument is again similar. Q.E.D.

COROLLARY 6.3. *Suppose* H *is a connected graph and for some* h, $\{h' : hIh'\}$ *is an independent set. Then* H *is a clique graph iff* $H - h$ *is.*

We make some remarks on identifying those graphs which have no independent cut sets in Sec. 6.

We next turn to another type of reduction which has proved useful in similar problems on interval graphs. (Cf., Roberts [7]). Let us say that two points h, h' of a graph H are <u>equivalent</u>, denoted hEh', if they belong to the same cliques. Thus hEh' holds iff $h = h'$ or hIh' and in addition h and h' are adjacent to exactly the same points $x \neq h, h'$. It is easy to verify that E is an equivalence relation. We define H^* to be the graph whose vertex set is the set of equivalence classes and so that two distinct equivalence classes are adjacent iff their representatives are adjacent.

<u>THEOREM 7.</u> <u>If</u> H^* <u>is a clique graph then</u> H <u>is</u>.

<u>Proof</u>. Suppose $\mathcal{K}^*$ is a collection of complete subgraphs of H^* satisfying (1) and (2) of Theorem 2. If $L^* \in \mathcal{K}^*$, let $L = \{h: [h] \in L^*\}$, where $[h]$ is the equivalence class containing h. It is easy to show each L is complete and $\mathcal{K} = \{L: L^* \in \mathcal{K}^*\}$ satisfies (1) and (2). Q.E.D.

<u>THEOREM 8.</u> <u>Suppose</u> $\omega(H) \leq 3$. <u>Then</u> H <u>is</u> <u>a clique graph iff</u> H^* <u>is</u>.

<u>Proof</u>. One direction follows by Theorem 7 and the other direction by the corollary to Theorem 4. To show the latter, note

that if H^* has a partial subgraph H_0 isomorphic to the graph of Fig. 2, then so does H; namely one whose vertices consist of one representative from each vertex in H_0. Q.E.D.

We have been unable to settle the question of whether the converse of Theorem 7 holds without any special assumptions about the clique number ω. To close this section we include a result which though not a reduction theorem itself, appears to be helpful in deriving reduction theorems.

Definition. We say a point g in a graph is simplicial, following Lekkerkerker and Boland [6], if $\{g' : gIg'\}$ is complete. We say a graph H is a strong clique graph if there is a G so that $H = K(G)$ and so that each clique of G contains a simplicial point. (Equivalently, each clique of G consists of a simplicial point and all its neighbors.)

THEOREM 9. *H is a clique graph if and only if it is a strong clique graph.*

Proof. If H is a strong clique graph it certainly is a clique graph. The converse follows by adding to each clique a point adjacent to all points in the clique. Formally, let $V(H) = \{h_1, h_2, \ldots, h_n\}$, let $H = K(G)$ with $V(G) = \{g_1, g_2, \ldots, g_m\}$, and let $K_1, K_2, \ldots, K_n$ be the cliques of G, so labelled that $h_i I h_j$ iff $K_i \cap K_j \neq \emptyset$. Then define G' by

$$\begin{cases} V(G') = \{g_1, g_2, \ldots, g_m, h_1, h_2, \ldots, h_n\} \\ g_i I' g_j \text{ iff } g_i I g_j \\ g_i I' h_j \text{ iff } g_i \in K_j \\ \text{not } h_i I' h_j\,. \end{cases}$$

Each h_i generates the clique $\{h_i\} \cup \{g_j : g_j \in K_i\}$. Any other clique K in G' would have to miss all the h_i. Therefore it would be a clique in G and so there would be an h_i adjacent to all the points of K in G'. Thus K would not be maximal and therefore would not be a clique. Q. E. D.

5. THE CONSECUTIVE ONES PROPERTY

Fulkerson and Gross [2] study a certain property of graphs which, in the light of Theorem 1, seems quite relevant to the clique graph notion. If H is a graph, Fulkerson and Gross study its (dominant) clique-vertex incidence matrix $\mathcal{M}(H) = (m_{ij})$. If $V(H) = \{h_1, h_2, \ldots, h_n\}$ and $L_1, L_2, \ldots, L_m$ are the cliques of H, then m_{ij} is defined by

$$m_{ij} = \begin{cases} 1 & \text{if } h_j \in L_i \\ 0 & \text{otherwise} \end{cases} .$$

In particular, Fulkerson and Gross study those graphs H whose matrices $\mathcal{M}(H)$ have the consecutive ones property (for columns), i.e., the rows may be permuted so that the ones in each column appear consecutively. It is easy to see that if $\mathcal{M}(H)$ satisfies the consecutive ones property, then $K(H)$ satisfies property $\mathcal{J}$. For suppose the cliques of H are listed in the order $L_1, L_2, \ldots, L_m$ so that the resulting matrix has ones in each column appearing consecutively. Suppose the collection $L_{i_1}, L_{i_2}, \ldots, L_{i_p}$, with $i_1 < i_2 < \ldots < i_p$, has pairwise intersections nonnull. Then there is an h_j in $L_{i_1} \cap L_{i_p}$. Since ones in each column appear consecutively, this h_j is in all the L_{i_r}. We have thus shown

THEOREM 16. If $\mathcal{M}(H)$ has the consecutive ones property, then H is a clique graph.

The converse of this theorem is false. For the collection of cliques of a square (a 4-cycle) has property $\mathcal{J}$, but the clique-vertex incidence matrix does not have the consecutive ones property. The following result now holds for the interval graphs which have been studied in Fulkerson and Gross [2], Gilmore and Hoffman [3], Lekkerkerker and Boland [6], and Roberts [7].

COROLLARY 10.1. Every interval graph is a clique graph.

Proof. By the results of Fulkerson and Gross, H is an interval graph iff $\mathcal{M}(H)$ has the consecutive ones property. Q.E.D.

6. GRAPHS WITH NO INDEPENDENT CUT SETS*

Corollary 6.2 suggests that it would be interesting to identify graphs with no independent cut sets. Our aim in this section is to present several reduction theorems relevant to this problem. We do not pretend to have solved it, but suggest it might be interesting and useful. The first reduction theorem concerns an interesting class of graphs which contains the interval graphs, the so-called rigid circuit graphs of Dirac [1].

THEOREM 11. *A rigid circuit graph has an independent cut set iff it has a cut point.*

Proof. By a theorem of Dirac, every minimal cut set in a rigid circuit graph is complete. Q.E.D.

Another helpful reduction result is the following.

THEOREM 12. *Suppose* H *is a graph and* h *is a vertex of* H *so that* $\{h': hIh'\}$ *is not independent. Then if* H–h *has no independent cut set, neither does* H.

Proof. Let S be an independent set in H and assume H–h has no independent cut sets.

Case 1. $h \in S$. Then if S were a cut set in H, S–h would be a cut set in H–h.

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*The results in this section were all obtained in conversations with Jon Folkman.

Case 2. $h \notin S$, Then S is not a cut set in H–h. Hence for all $x, y \in H-h$, $x, y \notin S$, there is a path in H–S–h from x to y. Hence there is such a path in H–S. Suppose now $x \neq h$, $x \notin S$. We show there is a path in H–S from x to h. Since $\{h' : hIh'\}$ is not independent, there is y such that hIy and $y \notin S$. We know that there is a path in H–S from x to y and hence there is one from x to y to h. Thus S is not a cut set in H. Q.E.D.

Dirac proves that every rigid circuit graph has a simplicial point, as defined above. (There are other graphs which have simplicial points as well.) The following reduction result is helpful for those graphs which have simplicial points.

THEOREM 13. *Suppose h is a simplicial point in a graph H and h has degree > 1. Then H has an independent cut set iff H–h has.*

Proof. If H–h has no independent cut set, then by the previous theorem, H has none either. Next assume that H has no independent cut set. Suppose S in H–h is an independent cut set in H–h. Then there are x, y in H–h so that there is no path from x to y in H–S–h. Next, S is not a cut set in H, so there is a (simple) path from x to y in H–S. It follows that the path goes through h and we may denote it $x, x_1, x_2, \ldots, x_n, h, y_1, y_2, \ldots, y_m, y$. Now x_n and y_1 are adjacent to h and so since h is simplicial, $x_n I y_1$. Thus there is a

path from x to y in H–S–h: $x, x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m, y$. This is a contradiction. Q. E. D.

This result together with the Dirac result that every rigid circuit graph has a simplicial point provides another fairly simple proof of Theorem 11.

REFERENCES

1. Dirac, G. A., "On Rigid Circuit Graphs," Abhandlungen Mathematischen Seminar Universität Hamburg, 25 (1961), 71–76.

2. Fulkerson, D. R., and O. A. Gross, "Incidence Matrices and Interval Graphs," Pacific J. Math., 15 (1965), 835–855.

3. Gilmore, P. C., and A. J. Hoffman, "A Characterization of Comparability Graphs and of Interval Graphs," Canad. J. Math., 16 (1964), 539–548.

4. Hamelink, R. C., "A Partial Characterization of Clique Graphs," J. of Combinatorial Theory 5 (1968), 192–197.

5. Krausz, J., "Demonstration nouvelle d'une theoreme de Whitney sur les reseaux," Mat. Fiz. Lapok 50 (1943), 75–85.

6. Lekkerkerker, C. G., and J. Ch. Boland, "Representation of a Finite Graph by a Set of Intervals on the Real Line," Fund. Math., 51 (1962), 45–64.

7. Roberts, F. S., "Indifference Graphs," In Proof Techniques in Graph Theory: Proceedings of the Second Ann Arbor Graph Conference, F. Harary (ed.), New York, Academic Press, 1969.

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10. ABSTRACT

A graph-theoretic discussion following up a recent paper in which Hamelink obtains an interesting sufficient condition for a graph to be a clique graph. In the present study related conditions are given that are necessary as well as sufficient. As an application of the result, it is shown that Hamelink's condition is also necessary in certain special cases and that here it can be greatly simplified. As another application, certain theorems are derived that are useful in practice in reducing the question of whether certain smaller or simpler graphs are clique graphs. Next, the clique graph results are related to some work of Fulkerson and Gross (RM-3984) on interval graphs and the consecutive 1's property. Finally, there are some remarks, motivated by the clique graph results, on graphs with no independent cut sets.

11. KEY WORDS

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